

Geometric Meanings of Curvatures in Finsler Geometry*

Zhongmin Shen

Dept. of Math., IUPUI, 402 N.Blackford Street, Indianapolis, IN 46202-3216, USA

1 Introduction

In Finsler geometry, we use calculus to study the geometry of regular inner metric spaces. In this note I will briefly discuss various curvatures and their geometric meanings from the metric geometry point of view, without going into the forest of tensors.

A metric d on a topological space M is a function on $M \times M$ with the following properties

(D1) $d(p, q) \geq 0$ and equality holds only when $p = q$;

(D2) $d(p, q) \leq d(p, r) + d(r, q)$.

For a Lipschitz curve $c : [a, b] \rightarrow (M, d)$, define the *dilation* of c at $t \in [a, b]$ by

$$\text{dil}_t(c) := \limsup_{\epsilon \rightarrow 0^+} \sup_{-\epsilon + t < t_1 < t_2 < t + \epsilon} \frac{d(c(t_1), c(t_2))}{t_2 - t_1}.$$

We obtain a length structure on M defined by

$$\ell_d(c) := \int_a^b \text{dil}_t(c) dt.$$

d is said to be *inner* if

$$d(p, q) = \inf_c \ell_d(c),$$

where the infimum is taken over all Lipschitz curves c from p to q . Traditionally, we impose the following reversibility condition on d

*This article is for the 20th Winter School on Geometry and Physics at Srni in Czech Republic. It was written during my visit at Institute of Mathematics and Informatics (IMI) at University of Debrecen in Hungary. The author would like to thank Dr. S. Bácsó and Dr. L. Kozma for their great help and hospitality.

$$(D3) \quad d(p, q) = d(q, p).$$

But this reversibility condition is so restrictive that it eliminates lots of interesting metric structures, such as the Funk metric below.

Let Ω be a strongly convex bounded domain in \mathbb{R}^n . For $p, q \in \Omega$, let ℓ_{pq} denote the ray issuing from p to q passing through q . Define

$$d_f(p, q) := \ln \frac{|z - p|}{|z - q|}, \quad (1)$$

where $z \in \partial\Omega$ is the intersection point of ℓ_{pq} with $\partial\Omega$. Then d_f is an inner metric on Ω , which is called the *Funk metric* [Funk]. The Funk metric is not reversible. Set

$$d_h(p, q) := \frac{1}{2} \left(d_f(p, q) + d_f(q, p) \right), \quad p, q \in \Omega. \quad (2)$$

We obtain a reversible inner metric which is called the *Hilbert metric*. There are many other interesting inner metrics which are not Riemannian.

An inner metric d on a manifold M is said to be *regular* if there is a non-negative function F on TM such that

$$(F0) \quad \text{for any } C^1 \text{ curve } c : [a, b] \rightarrow M, \operatorname{dil}_t(c) = F(\dot{c}(t)), \quad a \leq t \leq b;$$

$$(F1) \quad F \text{ is } C^\infty \text{ on } TM \setminus \{0\};$$

$$(F2) \quad \text{For each } x \in M, F_x := F|_{T_x M} \text{ is a Minkowski functional on } T_x M, \text{ i.e.,}$$

$$(F2a) \quad F_x(\lambda y) = \lambda F_x(y), \quad \forall \lambda > 0, y \in T_x M;$$

$$(F2b) \quad \text{for each } y \in T_x M \setminus \{0\}, \text{ the induced symmetric bilinear form } g_y \text{ on } T_x M \text{ is an inner product, where}$$

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[F^2(y + su + tv) \right]_{s=t=0}, \quad u, v \in T_x M. \quad (3)$$

A *Finsler metric* on a manifold M is a nonnegative function F on TM which satisfies (F1) and (F2).

The Funk metric d_f in (1) is regular and the induced Finsler metric F_f is determined by

$$x + \frac{y}{F_f(y)} \in \partial\Omega, \quad y \in T_x\Omega. \quad (4)$$

The Hilbert metric d_h in (2) is regular too and its induced Finsler metric F_h is determined by

$$F_h(y) := \frac{1}{2} \left(F_f(y) + F_f(-y) \right). \quad (5)$$

T. Okada [Ok] proved that the Funk metric F_f satisfies the following equation

$$\frac{\partial F_f}{\partial x^i} = F_f \frac{\partial F_f}{\partial y^i}. \quad (6)$$

Okada uses to prove the fact that F_f is of constant curvature $\kappa = -\frac{1}{4}$ and F_h is of constant curvature $\kappa = -1$.

2 Minkowski Spaces

Minkowski spaces are finite dimensional vector spaces equipped with a Finsler metric invariant under translations. Thus Minkowski spaces are just vector spaces equipped with Minkowski functionals. For a general Finsler space (M, F) , each tangent space $T_x M$ with $F_x := F|_{T_x M}$ is a Minkowski space. Thus to study the geometric structure of a Finsler space, we need to study Minkowski spaces first.

Let (V, F) be an n -dimensional Minkowski space. For each $y \in V \setminus \{0\}$, F induces an inner product g_y by (3). g_y satisfies the following homogeneity condition

$$g_{\lambda y}(u, v) = g_y(u, v), \quad \lambda > 0.$$

Note that g_y is independent of y if and only if F is Euclidean. It is natural to introduce the following quantity:

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [g_{y+tw}(u, v)] \Big|_{t=0}. \quad (7)$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in V \setminus \{0\}}$ is called the *Cartan torsion*. One can easily verify that \mathbf{C}_y is a symmetric multi-linear form on V . Moreover, \mathbf{C}_y satisfies the following homogeneity condition

$$\mathbf{C}_{\lambda y}(u, v, w) = \lambda^{-1} \mathbf{C}_y(u, v, w), \quad \lambda > 0.$$

Note that $\mathbf{C} = 0$ if and only if F is Euclidean. Differentiating \mathbf{C}_y with respect to y yields a new quantity:

$$\tilde{\mathbf{C}}_y(u, v, w, z) := \frac{d}{dt} [\mathbf{C}_{y+tz}(u, v, w)] \Big|_{t=0}. \quad (8)$$

Let $\tilde{\mathbf{C}} := \{\tilde{\mathbf{C}}_y\}_{y \in V \setminus \{0\}}$. $\tilde{\mathbf{C}}$ also gives us some geometric information on the Finsler metric [Sh1].

The mean of \mathbf{C}_y is defined by

$$\mathbf{I}_y(u) := \sum_{i,j=1}^n g^{ij}(y) \mathbf{C}_y(e_i, e_j, u), \quad (9)$$

where $g_{ij}(y) = g_y(e_i, e_j)$. The family $\mathbf{I} = \{\mathbf{I}_y\}_{y \in V \setminus \{0\}}$ is called the *mean Cartan torsion*. Deicke's Theorem [De] says that $\mathbf{C} = 0$ if and only if $\mathbf{I} = 0$. Note that in dimension two, the family $\mathbf{I} = \{\mathbf{I}_y\}_{y \in V \setminus \{0\}}$ completely determines the Cartan torsion.

There is another interesting quantity for Minkowski spaces associated with a Haar measure. Let μ be a Haar measure on V which is invariant under translations. Take an arbitrary basis $\{e_i\}_{i=1}^n$ for V and its dual basis $\{\omega^i\}_{i=1}^n$ for V^* , μ can be expressed by $d\mu = \sigma \omega^1 \wedge \cdots \wedge \omega^n$. We define

$$\tau(y) := \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma}, \quad (10)$$

where $g_{ij}(y) := g_y(e_i, e_j)$. τ is a well-defined quantity which is called the *distortion* of (F, μ) [Sh2][Sh3]. $\tau(y)$ satisfies the following homogeneity condition

$$\tau(\lambda y) = \tau(y), \quad \lambda > 0.$$

In general, $\tau(y)$ depends on the direction y . Differentiating $\tau(y)$ with respect to y yields the mean Cartan torsion.

$$\frac{d}{dt} [\tau(y + tv)] \Big|_{t=0} = \mathbf{I}_y(v). \quad (11)$$

Therefore, the following conditions are equivalent (a) $\tau(y) = \text{constant}$; (b) $\mathbf{I} = 0$; (c) $\mathbf{C} = 0$; (d) F is Euclidean.

There are several special Haar measures on a Minkowski space (V, F) . One of the natural Haar measures is the *Busemann-Hausdorff* measure μ_F . μ_F can be expressed by $d\mu_F = \sigma_F \omega^1 \wedge \cdots \wedge \omega^n$, where

$$\sigma_F := \frac{\text{Vol}(\mathbf{B}^n)}{\text{Vol}\{(y^i) \in \mathbf{R}^n, F(y^i e_i) < 1\}}, \quad (12)$$

where \mathbf{B}^n denote the unit ball in \mathbf{R}^n and Vol denotes the Euclidean measure on \mathbf{R}^n . The Busemann-Hausdorff measure is the unique Haar measure μ such that the unit ball \mathbf{B} in (V, F) has the same volume as the standard unit ball \mathbf{B}^n in \mathbf{R}^n . It is somewhat surprising that the Busemann-Hausdorff volume of the Funk metric F_f in (4) is finite. More precisely, for any metric r -ball $B(x, r)$ in the Funk space (Ω, F_f) ,

$$\mu_F(B(x, r)) = n \cdot 2^n \cdot \text{Vol}(\mathbf{B}^n) \int_0^{r/2} e^{-(n+1)t} \sinh^{n-1}(t) dt \rightarrow \text{Vol}(\mathbf{B}^n).$$

Let (V, F) be an n -dimensional Minkowski space and $\mathbf{S} = F^{-1}(1)$ the indicatrix. There are two induced metric structures on the indicatrix \mathbf{S} . One is

the Riemannian metric \dot{g} induced by g_y , and the other is the Finsler metric \dot{F} induced by F .

In 1949, L.A. Santaló [Sa] proved that if F is reversible, then the Riemannian volume of the indicatrix S satisfies

$$\mu_{\dot{g}}(S) \leq \text{Vol}(S^{n-1}), \quad (13)$$

equality holds if and only if F is Euclidean. However, there is no uniform lower bound on $\mu_{\dot{g}}(S)$. For the further study on the Minkowski functional F , one has to study the geometry of (S, \dot{g}) . It is surprising that the Riemannian curvature tensor $\dot{\mathbf{R}}_y$ of \dot{g} at $y \in S$ takes a special form as follows:

$$\dot{\mathbf{R}}_y(u, v)w = \mathbf{C}_y(\mathbf{C}_y(u, w), v) - \mathbf{C}_y(\mathbf{C}_y(v, w), u) + \dot{g}_y(v, w)u - \dot{g}_y(u, w)v, \quad (14)$$

where $u, v, w \in T_y S \subset V$ and $\mathbf{C}_y(u, v) = \xi$ is determined by $g_y(\xi, w) := \mathbf{C}_y(u, v, w)$. The Brickell theorem says that in dimension $n = \dim V > 2$, \dot{g} has constant curvature $\kappa = 1$ if and only if F is Euclidean [Bri].

For the Busemann-Hausdorff measure $\mu_{\dot{F}}$ on the indicatrix S , we have

$$c_n \leq \mu_{\dot{F}}(S) \leq c'_n, \quad (15)$$

where c_n and c'_n are positive constants depending only on n . No sharp constants have been determined in higher dimension. If F is non-reversible, however, there is no uniform upper bound on $\mu_{\dot{F}}(S)$. For further investigation on the Minkowski functional F , one has to study the geometry of (S, \dot{F}) . Suppose that \dot{F} is of constant curvature $\kappa = 1$. Is F Euclidean?

3 Connection and Geodesics

Now we consider general Finsler spaces. Geodesics are the first objects coming to a geometer's sight when he walks into an inner metric space. By definition, geodesics are locally length-minimizing constant speed curves which are characterized locally by a system of second order ordinary differential equations.

Let (M, F) be a Finsler space. For a C^1 curve $c : [a, b] \rightarrow M$, the length of c is given by

$$\ell(c) = \int_a^b F(\dot{c}(t))dt.$$

A direct computation yields the Euler-Lagrange equations for a geodesic $c(t)$

$$\frac{d^2 x^i}{dt^2} + 2G^i(\dot{c}) = 0, \quad (16)$$

where $(x^i(t))$ denote the coordinates of $c(t)$ and G^i in the standard local coordinate system (x^i, y^i) in TM are given by

$$G^i(y) := \frac{1}{4}g^{il}(y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y)\right\}y^j y^k. \quad (17)$$

where $g_{ij}(y) = g_y(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

A Finsler metric is said to be *positively complete* (resp. *complete*) if every geodesic on (a, b) can be extended to a geodesic defined on (a, ∞) (resp. $(-\infty, \infty)$). The Funk metric in (4) is positively complete, but not complete, while the Hilbert metric in (5) is complete. Finsler metrics on a compact manifold are always complete regardless the reversibility.

With the geodesic coefficients G^i in (17), we define a map $D_y : C^\infty(TM) \rightarrow T_x M$ for each $y \in T_x M$ by

$$D_y U := \left\{ dU^i(y) + U^j(x) \frac{\partial G^i}{\partial y^j}(y) \right\} \frac{\partial}{\partial x^i} \Big|_x,$$

where $U = U^i \frac{\partial}{\partial x^i} \in C^\infty(TM)$. $D_y U$ is called the *covariant derivative* of U in the direction y . We call the family $D := \{D_y\}_{y \in TM}$ the *canonical connection* of F . W. Barthel first noticed this canonical connection. With this connection D , we can define the covariant derivative $D_{\dot{c}} U(t)$ of a vector field $U(t)$ along a curve $c(t)$, $a \leq t \leq b$. $U(t)$ is said to be *parallel* along c if $D_{\dot{c}} U(t) = 0$. Clearly, a curve c is a geodesic if and only if the tangent vector field $\dot{c}(t)$ is parallel along c . The parallel translation $P_c : T_{c(a)} M \rightarrow T_{c(b)} M$ is defined by

$$P(U(a)) = U(b)$$

where $U(t)$ is parallel along c . From the definition, we see that P_c is a linear transformation preserving the inner products g_c . In general, P_c does not preserve the Minkowski functionals. We will discuss this issue in the next section.

It is natural to study the holonomy group defined by the above parallel translations. A natural question is whether or not there are more types of holonomy groups of Finsler spaces than the Riemannian case. This problem remains open so far.

Let (M, F) be a positively complete Finsler space. At each point $x \in M$, we define a map $\exp_x : T_x M \rightarrow M$ by

$$\exp_x(y) := c(1),$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$. The Hopf-Rinow theorem says that \exp_x is onto for all $x \in M$. \exp_x is called the *exponential map* at x . From the O.D.E. theory, J.H.C. Whitehead [Wh] proved that \exp_x is C^∞ on $T_x M \setminus \{0\}$ and only C^1 at the origin. Akbar-Zadeh [AZ] proved that \exp_x is C^2 at the origin for all x if and only if D is an affine connection.

4 Non-Riemannian Curvatures

The canonical connection D has all the properties of an affine connection except for the linearity in y . Namely, $D_{y_1+y_2} \neq D_{y_1} + D_{y_2}$ in general. To measure the non-linearity, it is natural to introduce the following quantity [Sh1]

$$\mathbf{B}_y(u, v, w) := \frac{\partial^2}{\partial s \partial t} \left[D_{y+sv+tw} U \right] \Big|_{s=t=0}, \quad (18)$$

where $U \in C^\infty(TM)$ with $U(x) = u$. One can easily verify that \mathbf{B}_y is a symmetric multi-linear form on $T_x M$. We call the family $\mathbf{B} := \{\mathbf{B}_y\}_{y \in TM \setminus \{0\}}$ the *Berwald curvature*. A Finsler metric is called a *Berwald metric* if $\mathbf{B} = 0$. L. Berwald proved a simple fact that $\mathbf{B} = 0$ if and only if D is an affine connection.

For Riemannian metrics, $\mathbf{B} = 0$ and D is just the Levi-Civita connection. There are non-Riemannian Berwald metrics with $\mathbf{B} = 0$. Consider the following type of Finsler metric:

$$F(y) := \alpha(y) + \beta(y), \quad (19)$$

where $\alpha(y) := \sqrt{g(y, y)}$ is a Riemannian metric and $\beta(y)$ is a 1-form with α -length $\|\beta\| < 1$. F is called a *Randers metric*. M Hashiguchi and Y. Ichijō [HaIc1] first noticed that if β is parallel with respect to α , then $F = \alpha + \beta$ is a Berwald metric. Later, they proved that if $d\beta = 0$, then $F = \alpha + \beta$ has the same geodesics as α and vice versa [HaIc2].

Y. Ichijō [Ic] proved that on a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals. Thus Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space. According to Szabó [Sz], if a Finsler metric F is Berwaldian, then there is a Riemannian metric g whose Levi-Civita connection coincides with the canonical connection of F .

Define the mean of \mathbf{B}_y by

$$\mathbf{E}_y(u, v) := \frac{1}{2} \sum_{i=1}^n g^{ij}(y) g_y(\mathbf{B}_y(u, v, e_i), e_j), \quad (20)$$

where $g_{ij}(y) = g_y(e_i, e_j)$. The family $\mathbf{E} = \{\mathbf{E}_y\}_{y \in TM \setminus \{0\}}$ is called the *mean Berwald curvature*. \mathbf{E} is also related to the S-curvature \mathbf{S} . See [Sh1] and (33) below.

As we have mentioned above, the parallel translation along curve in a Berwald space preserves the Minkowski functionals. Thus the Cartan torsion in a Berwald space does not change along geodesics. To measure the rate of changes of the Cartan torsion along geodesics in a general Finsler space, we will introduce a weaker quantity than the Berwald curvature. For a vector $y \in T_x M \setminus \{0\}$, let $c(t)$ denote the geodesic with $\dot{c}(0) = y$. Take arbitrary vectors $u, v, w \in T_x M$

and extend them to parallel vector fields $U(t), V(t), W(t)$ along c . Define

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt} \left[\mathbf{C}_{\dot{c}(t)}(U(t), V(t), W(t)) \right] \Big|_{t=0}. \quad (21)$$

The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM \setminus \{0\}}$ is called the *Landsberg curvature*. A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$ [Sh1]. Landsberg metrics form an important class of Finsler spaces. We have the following equation [Sh1]

$$\mathbf{L}_y(u, v, w) = -\frac{1}{2} g_y(\mathbf{B}_y(u, v, w), y). \quad (22)$$

From (22), we immediately conclude that every Berwald space is a Landsberg space. It is an open problem in Finsler geometry whether or not there is a Landsberg metric which is not a Berwald metric. So far no example has been found. Differentiating \mathbf{L} along geodesics yields a new quantity:

$$\dot{\mathbf{L}}_y(u, v, w) := \frac{d}{dt} \left[\mathbf{L}_{\dot{c}(t)}(U(t), V(t), W(t)) \right] \Big|_{t=0}. \quad (23)$$

Using (6), we can show that the Funk metric $F = F_f$ in (4) satisfies

$$\mathbf{L}_y(u, v, w) + \frac{1}{2} F(y) \mathbf{C}_y(u, v, w) = 0, \quad (24)$$

and the Hilbert metric in (5) satisfies

$$\dot{\mathbf{L}}_y(u, v, w) - F^2(y) \mathbf{C}_y(u, v, w) = 0. \quad (25)$$

The Landsberg curvature \mathbf{L}_y satisfies the following homogeneity condition

$$\mathbf{L}_{\lambda y}(u, v, w) = \mathbf{L}_y(u, v, w), \quad \lambda > 0. \quad (26)$$

In general, \mathbf{L}_y depends on the direction y . Differentiating \mathbf{L}_y with respect to y yields another quantity [Sh1]

$$\tilde{\mathbf{L}}_y(u, v, w, z) := \frac{d}{dt} \left[\mathbf{L}_{y+tz}(u, v, w) \right] \Big|_{t=0}. \quad (27)$$

One can easily verify that $\tilde{\mathbf{L}} = 0$ if and only if $\mathbf{L} = 0$. When $\mathbf{L} \neq 0$, $\tilde{\mathbf{L}}$ gives us some other geometric information on the Finsler metric.

Define the mean of \mathbf{L}_y by

$$\mathbf{J}_y(u) := \sum_{ij=1}^n g^{ij}(y) \mathbf{L}_y(u, e_i, e_j). \quad (28)$$

The family $\mathbf{J} = \{\mathbf{J}_y\}_{y \in TM \setminus \{0\}}$ is called the *mean Landsberg curvature* [Sh1]. From the definitions of \mathbf{I} and \mathbf{J} , we have

$$\mathbf{J}_y(u) = \frac{d}{dt} \left[\mathbf{I}_{\dot{c}(t)}(U(t)) \right] \Big|_{t=0}, \quad (29)$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$ and $U(t)$ is a parallel vector field along c with $U(0) = u$. In dimension two, \mathbf{J} completely determines \mathbf{L} . It is an interesting problem to study the difference between Finsler metrics with $\mathbf{J} = 0$ and those with $\mathbf{L} = 0$.

There is an induced Riemannian metric of Sasaki type on $TM \setminus \{0\}$. T. Aikou proved that if $\mathbf{L} = 0$, then all the slit tangent spaces $T_x M \setminus \{0\}$ are totally geodesic in $TM \setminus \{0\}$ [Ai]. Along the same line, one can show that if $\mathbf{J} = 0$, then all the slit tangent spaces $T_x M \setminus \{0\}$ are minimal in $TM \setminus \{0\}$.

Consider an arbitrary regular measure μ on a Finsler space (M, F) . μ induces a Haar measure μ_x in each tangent space $T_x M$. Hence the distortion τ is defined for $(T_x M, F_x, \mu_x)$. To measure the rate of changes of the distortion along geodesics, we define

$$\mathbf{S}(y) := \frac{d}{dt} \left[\tau(\dot{c}(t)) \right] \Big|_{t=0}, \quad (30)$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$. We call the scalar function \mathbf{S} the *S-curvature* [Sh1][Sh3]. Differentiating the S-curvature along geodesics yields a new quantity:

$$\dot{\mathbf{S}}(y) := \frac{d}{dt} \left[\mathbf{S}(\dot{c}(t)) \right] \Big|_{t=0}. \quad (31)$$

See [Sh2] for further discussions. When $\mathbf{S} \neq 0$, $\dot{\mathbf{S}}$ gives us some other geometric information on the Finsler metric F and the regular measure μ . See (37) below.

The S-curvature $\mathbf{S}(y)$ satisfies the following homogeneity condition

$$\mathbf{S}(\lambda y) = \lambda \mathbf{S}(y), \quad \lambda > 0. \quad (32)$$

In general, $\mathbf{S}(y)$ is not linear in y . Differentiating it twice with respect to y gives no new quantity. Namely, we have

$$\mathbf{E}_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[\mathbf{S}(y + su + tv) \right] \Big|_{s=t=0}. \quad (33)$$

Thus $\mathbf{S}(y)$ is linear in $y \in T_x M$ if and only if $\mathbf{E} = 0$ on $T_x M \setminus \{0\}$. In particular, if F is a Berwald metric, then $\mathbf{S}(y)$ is linear in $y \in T_x M$ for all x [Sh1]. In fact, $\mathbf{S} = 0$ for Berwald metrics if we consider the S-curvature of the Busemann-Hausdorff measure μ_F . This fact is proved by the author [Sh3]. Finsler spaces with $\mathbf{E} = 0$

deserve further investigation. There are some non-Berwaldian Randers metrics with $\mathbf{E} = 0$ and $\mathbf{S} = 0$. For the Funk metric $F = F_f$ in (4), the S-curvature and the mean Berwald curvature are constant in the following sense.

$$\begin{aligned}\mathbf{S}(y) &= \frac{n+1}{2}F(y), \\ \mathbf{E}_y(u, v) &= \frac{n+1}{4F^3(y)} \left\{ F^2(y)g_y(u, v) - g_y(y, u)g_y(y, v) \right\}.\end{aligned}$$

This is proved in [Sh1].

5 Riemann Curvature

As matter of fact, all the quantities defined in the previous sections vanish on a Riemannian space. Thus we do not see these non-Riemannian quantities at all in Riemannian geometry. A. Einstein used Riemannian geometry to describe his general relativity theory, assuming that a spacetime is *always* Riemannian.

For Riemannian spaces, there is only one notion of curvature—Riemann curvature, that was introduced by B. Riemann in 1854 as a generalization of the Gauss curvature for surfaces. Since then, the Riemann curvature became the central concept in Riemannian geometry. Due to the efforts by L. Berwald in 1920's, the Riemann curvature can be extended to the Finslerian case [Ber].

Let (M, g) be a Riemannian space and D denote the Levi-Civita connection of g . The *Riemann curvature tensor* is defined by

$$R(u, v)w := \left\{ D_U D_V W - D_V D_U W - D_{[U, V]} W \right\}|_x, \quad u, v, w \in T_x M,$$

where U, V, W are local vector fields with $U(x) = u, V(x) = v, W(x) = w$. The core part of the Riemann curvature tensor is the following quantity:

$$\mathbf{R}_y(u) := R(u, y)y. \quad (34)$$

The Riemann curvature $\mathbf{R}_y : T_x M \rightarrow T_x M$ is a self-adjoint linear transformation with respect to g and it satisfies $\mathbf{R}_y(y) = 0$. The family $\mathbf{R} = \{\mathbf{R}_y\}_{y \in TM \setminus \{0\}}$ is called the *Riemann curvature*. With a little trick by the author, one can extend the notion of Riemann curvature to Finsler metrics without employing connections on the slit tangent bundle $TM \setminus \{0\}$.

Let (M, F) be a Finsler space. Given a vector $y \in T_x M \setminus \{0\}$, extend it to a local nowhere zero *geodesic field* Y (i.e., all integral curves of Y are geodesics). Y induces a Riemannian metric

$$\hat{g} := g_Y.$$

Let $\hat{\mathbf{R}}$ denote the Riemann curvature of \hat{g} as defined above. Define

$$\mathbf{R}_y := \hat{\mathbf{R}}_y. \quad (35)$$

One can verify that \mathbf{R}_y is independent of the geodesic extension Y of y . Moreover, \mathbf{R}_y is self-adjoint with respect to g_y , i.e.,

$$g_y(\mathbf{R}_y(u), v) = g_y(u, \mathbf{R}_y(v)),$$

and it satisfies $\mathbf{R}_y(y) = 0$ [Sh1][Sh2]. Let $W_y := \{u \in T_x M, g_y(y, u) = 0\}$. Then $\mathbf{R}_y|_{W_y} : W_y \rightarrow W_y$ is again a self-adjoint linear transformation with respect to g_y . Denote the eigenvalues of $\mathbf{R}_y|_{W_y}$ by

$$\kappa_1(y) \leq \dots \leq \kappa_{n-1}(y).$$

They are the most important intrinsic invariants of the Finsler metric. We call $\kappa_i(y)$ the i -th principal curvature in the direction y . The trace of \mathbf{R}_y is denoted by $\mathbf{Ric}(y)$ which is called the *Ricci curvature*. $\mathbf{Ric}(y)$ is given by

$$\mathbf{Ric}(y) := \sum_{i,j=1}^n g^{ij}(y) g_y(\mathbf{R}_y(e_i), e_j) = \sum_{i=1}^{n-1} \kappa_i(y). \quad (36)$$

The Ricci curvature and the S-curvature determine the local behavior of the Busemann-Hausdorff measure of small metric balls around a point. Let B_x denote the unit ball in $(T_x M, F_x)$ and μ_x the induced Busemann-Hausdorff measure of F_x on $T_x M$. Assume that F is reversible. Then the Taylor expansion of $\mu_F(B(x, \epsilon))$ of a small metric ball $B(x, \epsilon)$ is given by

$$\mu_F(B(x, \epsilon)) = \text{Vol}(B^n) \left\{ 1 - \frac{1}{6(n+2)} r(x) \epsilon^2 + O(\epsilon^3) \right\}, \quad (37)$$

where

$$r(x) := \frac{n+2}{n \cdot \text{Vol}(B^n)} \int_{B_x} \left\{ \mathbf{Ric}(y) d\mu_x + 3n [\dot{\mathbf{S}}(y) - \mathbf{S}^2(y)] \right\} d\mu_x. \quad (38)$$

See [Sh2] for details.

6 Constant Curvature

Now let us take a close look at Finsler spaces of constant curvature κ . A Finsler metric is said to be of *scalar curvature* if there is a scalar function $\kappa(y)$ on $TM \setminus \{0\}$ such that for any $y \in T_x M \setminus \{0\}$, the principal curvatures $\kappa_i(y) = \kappa(y)$, $i = 1, \dots, n-1$. By definition, all two dimensional Finsler metrics are of scalar curvature $\kappa(y)$. F is said to be of *constant curvature* κ (resp. *constant Ricci curvature*) if $\kappa_i(y) = \kappa$, $i = 1, \dots, n-1$ (resp. $\sum_{i=1}^{n-1} \kappa_i(y) = (n-1)\kappa$).

We have the following important equation [AZ]

$$\dot{\mathbf{L}}_y(u, v, w) + \kappa F^2(y) \mathbf{C}_y(u, v, w) = 0. \quad (39)$$

The Cartan torsion and Landsberg curvature take special values along geodesics. Let $c(t)$ be an arbitrary *unit speed* curve. Take a parallel vector field $V(t)$ along $c(t)$. Let

$$\mathbf{C}(t) := \mathbf{C}_{\dot{c}(t)}(V(t), V(t), V(t)). \quad (40)$$

From (23) and (39), we obtain the following important equation [Nu][AZ]

$$\mathbf{C}''(t) + \kappa \mathbf{C}(t) = 0. \quad (41)$$

This immediately implies that Landsberg space of constant curvature $\kappa \neq 0$ must be Riemannian. This is observed by S. Numata [Nu]. Solving (41), we obtain

$$\mathbf{C}(t) = \begin{cases} a \sinh(t) + b \cosh(t), & \text{if } \kappa = -1, \\ at + b, & \text{if } \kappa = 0, \\ a \sin(t) + b \cos(t), & \text{if } \kappa = 1. \end{cases} \quad (42)$$

Define $\mathbf{L}(t)$ in the same way as above for the Landsberg curvature. From the definition of \mathbf{L} , we have $\mathbf{L}(t) = \mathbf{C}'(t)$. Then we obtain a formula for $\mathbf{L}(t)$ [AZ].

Take two parallel vector fields $V(t)$ and $W(t)$ along c . Assume that both $V(t)$ and $W(t)$ are $g_{\dot{c}(t)}$ -orthogonal to $\dot{c}(t)$ for some $t = t_o$ (hence for all t). Set

$$\tilde{\mathbf{C}}(t) := \tilde{\mathbf{C}}_{\dot{c}(t)}(V(t), V(t), V(t), W(t)).$$

By studying the Ricci identities and the Bianchi identities, we obtain

$$\tilde{\mathbf{C}}(t) = \begin{cases} a \sinh(2t) + b \cosh(2t) + c, & \text{if } \kappa = -1, \\ at^2 + bt + c, & \text{if } \kappa = 0, \\ a \sin(2t) + b \cos(2t) + c, & \text{if } \kappa = 1. \end{cases} \quad (43)$$

Define $\tilde{\mathbf{L}}(t)$ in the same way as above for $\tilde{\mathbf{L}}$. We can show that $\tilde{\mathbf{L}}(t) = \tilde{\mathbf{C}}'(t) + c'$. Then we obtain a formula for $\tilde{\mathbf{L}}(t)$ [Sh1].

Complete Finsler metrics of constant curvature $\kappa < 0$ must be Riemannian if the Cartan torsion does not grow exponentially. This fact is due to Akbar-Zadeh [AZ]. Using (6), T. Okada [Ok] verified that the Funk metric F_f in (4) is of constant curvature $\kappa = -\frac{1}{4}$ and the Hilbert metric F_h in (5) is of constant curvature $\kappa = -1$. By (24), we can show that the Cartan torsion of F_f is bounded along any geodesic. Note that F_f is not Riemannian ! because it is only positively complete. Since F_h is non-Riemannian, the Cartan torsion of F_h must grow exponentially along geodesics in one direction.

Positively complete Finsler spaces of constant curvature $\kappa = 0$ must be locally Minkowski if \mathbf{C} and $\tilde{\mathbf{C}}$ are bounded along geodesics. This fact is also due to Akbar-Zadeh [AZ]. So far, we do not know if there are any positively complete Finsler spaces of constant curvature $\kappa = 0$, except for locally Minkowski spaces.

There are infinitely many projectively flat Finsler metrics of constant curvature $\kappa = 1$ on S^n constructed by R. Bryant [Br1][Br2] recently. Bryant metrics are non-reversible. So far, no reversible Finsler metric of constant curvature $\kappa = 1$ has been found on S^n , except for the standard Riemannian metric. The author can prove that for any Finsler metric on a simply connected compact manifold M , if it has constant curvature $\kappa = 1$, then M must be diffeomorphic to S^n and geodesics are all closed with length of 2π . From (42) and (43), we see that \mathbf{C} has period of 2π on parallel vector fields along any geodesic, while $\tilde{\mathbf{C}}$ has period of π on parallel vector fields orthogonal to the geodesic [Sh1].

All known Finsler metrics of constant curvature are locally projectively flat, i.e., at every point, there is a local coordinate system in which the geodesics are straight lines. It is an interesting problem to find Finsler metrics of constant curvature without this property.

Consider two pointwise projectively related Finsler metrics F and \tilde{F} on a manifold. Suppose that F and \tilde{F} has constant Ricci curvature κ and $\tilde{\kappa}$, respectively. Then using A. Rapcsák's equation, we can show that for any unit speed geodesic $c(t)$ of F , the function $\varphi(t) := 1/\sqrt{\tilde{F}(\dot{c}(t))}$ satisfies

$$\varphi''(t) + \kappa\varphi(t) = \frac{\tilde{\kappa}}{\varphi^3(t)}. \quad (44)$$

See [Sh1][Sh4]. By (44), we can show that the Hilbert metric is the only complete, reversible, projectively flat Finsler metric of constant curvature $\kappa = -1$ on a strongly convex domain in \mathbb{R}^n . There are might be many positively complete non-reversible projectively flat Finsler metrics of constant curvature $\kappa = -\frac{1}{4}$ on a strongly convex domain in \mathbb{R}^n . So far we only have the Funk metric with this property.

It is an open problem whether or not there is a (positively) complete Finsler space which does not admit any (positively) complete Finsler metrics of scalar curvature. This leads to the study on the topology of (positively) complete Finsler spaces of scalar curvature.

7 Comparison Geometry

In this section, we will discuss several global results using comparison techniques.

Let (M, F) be a positively complete Finsler space. Take a geodesic variation $c_s(t)$ of a geodesic $c(t)$, i.e., $c_0(t) = c(t)$ and each $c_s(t)$ is a geodesic. Let $J(t) := \frac{\partial c_s}{\partial s}|_{s=0}(t)$. $J(t)$ is a vector field along c which is called a *Jacobi field*. The behavior of $J(t)$ along c is controlled by the following Jacobi equation

$$D_{\dot{c}}D_{\dot{c}}J(t) + \mathbf{R}_{\dot{c}(t)}(J(t)) = 0. \quad (45)$$

Take a geodesic $c(t) = \exp_x(ty)$, $0 \leq t < \infty$ and a special geodesic variation $c_s(t) := \exp_x(t(y + sv))$. The standard comparison argument by Cartan-Hadamard and Bonnet-Meyers gives the following important global results in comparison Finsler geometry.

Theorem 7.1 ([Aus]) *Let (M, F) be a positively complete Finsler space. Suppose that the Riemann curvature is nonpositive, i.e., the principal curvatures*

$$\kappa_i(y) \leq 0, \quad i = 1, \dots, n-1.$$

Then the exponential map $\exp_x : T_x M \rightarrow M$ is an onto covering map. Thus M is a $K(\pi, 1)$ space.

Theorem 7.2 ([Aus]) *Let (M, F) be a positively complete Finsler space. Suppose that the Ricci curvature is strictly positive, i.e., there is a positive constant λ such that*

$$\sum_{i=1}^{n-1} \kappa_i(y) \geq (n-1)\lambda > 0.$$

Then the exponential map $\exp_x : T_x M \rightarrow M$ is singular at ry for any unit vector $y \in T_x M$ at $r \leq \pi/\sqrt{\lambda}$. Thus the diameter of M and its universal cover \tilde{M} is bounded by $\text{Diam}(M) \leq \pi/\sqrt{\lambda}$, and the fundamental group $\pi_1(M)$ must be finite.

Applying the Morse theory to the loop space, one can prove the following theorem for homotopy groups.

Theorem 7.3 *Let (M, F) be a compact simply connected Finsler space. Suppose that the principal curvature $\kappa_1(y) \leq \dots \leq \kappa_{n-1}(y)$ satisfy the following pinching condition for some $2 \leq k \leq n-2$,*

$$\frac{1}{4} < \frac{1}{k} \sum_{i=1}^k \kappa_i(y), \quad \kappa_{n-1}(y) \leq 1. \quad (46)$$

Then $\pi_i(M) = 0$ for $i = 1, \dots, n-k$.

Let (M, F) be a positively complete space. Define by $B(x, r)$ and $S(x, r)$ the metric ball and sphere around x with radius r , respectively. There is a naturally induced measure ν_F on the regular part of $S(x, r)$ such that the coarea formula holds

$$\mu_F(B(x, r)) = \int_0^r \nu_F(S(x, t)) dt. \quad (47)$$

Let $\mu_{\tilde{F}}$ denote the Busemann-Hausdorff measure of the induced Finsler metric \tilde{F} on $S(x, r)$. In general, $\nu_F \neq \mu_{\tilde{F}}$. If F is reversible, then

$$c_n \mu_{\tilde{F}} \leq \nu_F \leq c'_n \mu_{\tilde{F}}, \quad (48)$$

where c_n, c'_n are positive constants. If F is non-reversible, the inequality on the left side of (48) does not hold. The coarea formula (47) together with (48) implies (15). See [Sh2] for more details. Further estimates on the geometry of $S(x, r)$ give the following comparison result on the Busemann-Hausdorff measure μ_F under certain curvature bounds.

Theorem 7.4 ([Sh2][Sh3]) *Let (M, F) be an n -dimensional positively complete Finsler space. Suppose that the Ricci curvature and the S -curvature satisfy*

$$\mathbf{Ric}/F^2 \geq (n-1)\lambda, \quad \mathbf{S}/F \geq (n-1)\delta, \quad (49)$$

where λ, δ are positive constants. Then the ratios $\mu_F(B(x, r))/V_{\lambda, \delta}(r)$ and $\nu_F(S(x, r))/V'_{\lambda, \delta}(r)$ are non-increasing, where

$$V_{\lambda, \delta}(r) := \text{Vol}(S^{n-1}) \int_0^r \left[e^{-\delta t} \mathbf{s}_\lambda(t) \right]^{n-1} dt,$$

and $\mathbf{s}_\lambda(t)$ satisfies

$$\mathbf{s}_\lambda''(t) + \lambda \mathbf{s}_\lambda(t) = 0, \quad \mathbf{s}_\lambda(0) = 0, \quad \mathbf{s}_\lambda'(0) = 1.$$

Theorem 7.4 has a number of applications. Let M be a compact oriented manifold. The canonical L^1 -norm $\|\cdot\|_1$ on the complex $C_k(M)$ of singular real chains is defined by

$$\|c\|_1 := \sum_i |r_i|, \quad c = \sum_i r_i \sigma_i.$$

For a real homology class $z \in H_k(M)$, define

$$\|z\|_1 = \inf_{z=[c]} \|c\|_1.$$

For the fundamental class $[M] \in H_n(M)$, let

$$\|M\| := \|[M]\|_1.$$

$\|M\|$ is called the *Gromov invariant* of M . $\|M\|$ is not necessarily an integer. Gromov proved that if $\pi_1(M)$ is amenable, then $\|M\| = 0$.

Theorem 7.5 *Let (M, F) be an n -dimensional reversible compact Finsler space. Suppose that the Ricci curvature and the S -curvature satisfy the bounds (49) with $\lambda, \delta \leq 0$. Then*

$$\|M\| \leq n!(n-1)^n (\sqrt{|\lambda|} + |\delta|)^n \mu_F(M). \quad (50)$$

Further, there is a constant $\epsilon(n) > 0$ if

$$(\sqrt{|\lambda|} + |\delta|)^n \mu_F(M) \leq \epsilon(n), \quad (51)$$

then $\|M\| = 0$.

The theorem for Riemannian spaces was proved by M. Gromov [Gr]. The proof for the general case follows from Gromov's argument by using Theorem 7.4.

References

- [Ai] T. Aikou, *Some remarks on the geometry of tangent bundles of Finsler spaces*, Tensor, N. S. **52**(1993), 234-242.
- [Aus] L. Auslander, *On curvature in Finsler geometry*, Trans. Amer. Math. Soc. **79**(1955), 378-388.
- [AZ] H. Akbar-Zadeh, *Sur les espaces de Finsler á courbures sectionnelles constantes*, Bull. Acad. Roy. Bel. Cl, Sci, 5e Série - Tome LXXXIV (1988) 281-322.
- [Ber] L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Z. **25**(1926), 40-73.
- [Bri] F. Brickell, *A theorem on homogeneous functions*, J. London Math. Soc. **42**(1967), 325-329.
- [Br1] R. Bryant, *Finsler structures on the 2-sphere satisfying $K = 1$* , Finsler Geometry, Contemporary Mathematics **196**, Amer. Math. Soc., Providence, RI, 1996, 27-42.
- [Br2] R. Bryant, *Projectively flat Finsler 2-spheres of constant curvature*, Selecta Math., New Series, **3**(1997), 161-204.
- [De] A. Deicke, *Über die Finsler-Räume mit $A_i = 0$* , Arch. Math. **4**(1953), 45-51.
- [Funk] P. Funk, *Über Geometrien, bei denen die Geraden die Kürzesten sind*, Math. Ann. **101**(1929), 226-237.
- [Gr] M. Gromov, *Volume and bounded cohomology*, I. H. E. S. Publ. Math. **56**(1983), 213-307.
- [HaIc1] M. Hashiguchi and Y. Ichijyō, *On some special (α, β) metrics*, Rep. Fac. Sci. Kagoshima Univ. **8**(1975), 39-46.
- [HaIc2] M. Hashiguchi and Y. Ichijyō, *Randers spaces with rectilinear geodesics*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. & Chem.), **13**(1980), 33-40.
- [Ic] Y. Ichijyō, *Finsler spaces modeled on a Minkowski space*, J. Math. Kyoto Univ. **16**(1976), 639-652.
- [Nu] S. Numata, *On Landsberg spaces of scalar curvature*, J. Korea Math. Soc. **12**(1975), 97-100.

- [Ok] T. Okada, *On models of projectively flat Finsler spaces of constant negative curvature*, Tensor, N. S. **40**(1983), 117-123.
- [Sa] L.A. Santalò, *Un invariante afin para los cuerpos convexos del espacio de n dimensiones*, Portugaliae Mathematica, **8** (1949), 154-161.
- [Sh1] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.
- [Sh2] Z. Shen, *Lectures on Finsler Geometry*, in preparation.
- [Sh3] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances in Math. **128**(1997), 306-328.
- [Sh4] Z. Shen, *On projectively related Einstein metrics in Riemann-Finsler geometry*, to appear in Math. Ann.
- [Sz] Z. Szabó, *Positive definite Berwald spaces (Structure theorems on Berwald spaces)*, Tensor, N. S. **35**(1981), 25-39.
- [Wh] J.H.C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. Oxford Ser. **3**(1932), 33-42.

zshen@math.iupui.edu